Symmetry classification of Newtonian incompressible fluid's equations flow in turbulent boundary layers

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Abstract

Lie symmetry group method is applied to study Newtonian incompressible fluid's equations flow in turbulent boundary layers. The symmetry group and its optimal system are given, and group invariant solutions associated to the symmetries are obtained. Finally the structure of the Lie algebra symmetries is determined.

Key words: Fluid mechanics; Lie symmetry; Partial differential equation; Shear stress; Optimal system.

1 Introduction

In physics and fluid mechanics, a *boundary layer* is that layer of fluid in the immediate vicinity of a bounding surface. In the Earth's atmosphere, the planetary boundary layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding non-viscous flow. It is a phenomenon of viscous forces. This effect is related to the Reynolds number (In fluid mechanics and heat transfer, the *Reynold's number* is a dimensionless number that gives a measure of the ratio of inertial forces to viscous and, consequently, it quantifies the relative importance of these two types of forces for given flow conditions). Laminar boundary layers come in various forms and can be loosely classified according to their structure and the circumstances under which they are created. The thin shear layer which develops on an oscillating body is an example of a Stokes boundary layer, whilst the Blasius boundary layer refers to the well-known similarity solution for the steady boundary layer attached to a flat plate held in an oncoming unidirectional flow. When a fluid rotates, viscous forces may be balanced by the Coriolis effect, rather than convective inertia, leading to the formation of an *Ekman* layer. Thermal boundary layers also exist in heat transfer. Multiple types of boundary layers can coexist near a surface simultaneously. The deduction of the boundary layer equations was perhaps one of the most important advances in fluid dynamics. Using an order of magnitude analysis, the well-known governing *Navier−Stokes* equations of viscous fluid flow can be greatly simplified within the boundary layer. Notably, the characteristic of the partial differential equations (PDE) becomes parabolic, rather than the elliptical form of the full Navier−Stokes equations. This greatly simplifies the solution of the equations. By making the boundary layer approximation, the flow is divided into an inviscid portion (which is easy to solve by a number of methods) and the boundary layer, which is governed by an easier to solve PDE.

Flow and heat transfer of an incompressible viscous fluid over a stretching sheet appear in several manufacturing processes of industry such as the extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets, etc. In the glass industry, blowing, floating or spinning of fibres are processes, which involve the flow due to a stretching surface. Mahapatra and Gupta studied the steady two-dimensional stagnation-point flow of an incompressible viscous fluid over a flat deformable sheet when the sheet is stretched in its own plane with a

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velocity proportional to the distance from the stagnation-point. They concluded that, for a fluid of small kinematic viscosity, a boundary layer is formed when the stretching velocity is less than the free stream velocity and an inverted boundary layer is formed when the stretching velocity exceeds the free stream velocity. Temperature distribution in the boundary layer is determined when the surface is held at constant temperature giving the so called surface heat flux. In their analysis, they used the finite-differences scheme along with the *Thomas algorithm* to solve the resulting system of ordinary differential equations.

The treatment of turbulent boundary layers is far more difficult due to the time-dependent variation of the flow properties. One of the most widely used techniques in which turbulent flows are tackled is to apply *Reynolds decomposition*. Here the instantaneous flow properties are decomposed into a mean and fluctuating component. Applying this technique to the boundary layer equations gives the full turbulent boundary layer equations

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial x}+\dfrac{\partial \bar{v}}{\partial y}=0, \\ \bar{u}\dfrac{\partial \bar{u}}{\partial x}+\bar{v}\dfrac{\partial \bar{u}}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x}+\nu\Big(\dfrac{\partial^2 \bar{u}}{\partial x^2}+\dfrac{\partial^2 \bar{u}}{\partial y^2}\Big)-\dfrac{\partial}{\partial y}(\overline{u'v'})-\dfrac{\partial}{\partial x}(\overline{u'^2}), \\ \bar{u}\dfrac{\partial \bar{v}}{\partial x}+\bar{v}\dfrac{\partial \bar{v}}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial y}+\nu\Big(\dfrac{\partial^2 \bar{v}}{\partial x^2}+\dfrac{\partial^2 \bar{v}}{\partial y^2}\Big)-\dfrac{\partial}{\partial x}(\overline{u'v'})-\dfrac{\partial}{\partial y}(\overline{v'^2}), \end{cases} \tag{1}$$

where ρ is the *density*, p is the *pressure*, ν is the *kinematic viscosity* of the fluid at a point and $\bar{u}$ and $\bar{v}$ are average of the velocity components in Reynold decomposition. Here u' and v' are the *velocity fluctations* such that; $u=\bar{u}+u'$ and $v=\bar{v}+v'$. By using the *scale analysis*(a powerful tool used in the mathematical sciences for the simplification of equations with many terms), it can be shown that the system (1) reduce to the classical form

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial x}+\dfrac{\partial \bar{v}}{\partial y}=0, \\ \bar{u}\dfrac{\partial \bar{u}}{\partial x}+\bar{v}\dfrac{\partial \bar{v}}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x}+\nu\dfrac{\partial^2 \bar{u}}{\partial y^2}-\dfrac{\partial}{\partial y}(\overline{u'v'}), \\ \dfrac{\partial \bar{p}}{\partial y}=0. \end{cases} \tag{2}$$

The term $\overline{u'v'}$ in the system (2) called *Reynolds shear stress*, a tensor that conventionally written

$$R_{ij}\equiv\rho\overline{u_i'u_j'}. \tag{3}$$

The divergence of this stress is the force density on the fluid due to the *turbulent fluctuations*. Using NavierStokes equations for a fluid whose stress versus rate of strain curve is linear and passes through the origin (Newtonian fluid) the tensor (3) reduces to

$$R_{ij}\equiv\mu\frac{\partial \bar{u}_i}{\partial x_j},$$

where μ is the *fluid viscosity*.

This paper is concerned with the symmetry of Newtonian incompressible fluid's equations flow in turbulent boundary layers of the form (2). Lie-group theory is applied to the equations of motion for determining symmetry reductions of partial differential equations. The solution of the turbulent boundary layer equations therefore necessitates the use of a turbulence model, which aims to express the Reynolds shear stress in terms of known flow variables or derivatives. The lack of accuracy and generality of such models is a major obstacle in the successful prediction of turbulent flow properties in modern fluid dynamics

2 Lie Symmetries of the Equations

A PDE with $p-$independent and $q-$dependent variables has a Lie point transformations

$$\widetilde{x}_i=x_i+\varepsilon\xi_i(x,u)+\mathcal{O}(\varepsilon^2), \qquad \widetilde{u}_\alpha=u_\alpha+\varepsilon\varphi_\alpha(x,u)+\mathcal{O}(\varepsilon^2),$$

where $\xi_i = \frac{\partial \widetilde{x}_i}{\partial \varepsilon}\Big|_{\varepsilon=0}$ for $i = 1, ..., p$ and $\varphi_\alpha = \frac{\partial \widetilde{u}_\alpha}{\partial \varepsilon}\Big|_{\varepsilon=0}$ for $\alpha = 1, ..., q$. The action of the Lie group can be considered by its associated infinitesimal generator

$$\mathbf{v} = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{4}$$

on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (4) is given by

$$Q^\alpha(x,u^{(1)}) = \varphi_\alpha(x,u) - \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial u^\alpha}{\partial x_i},$$

and its $n-$th prolongation is determined by

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{\sharp J=j=0}^{n} \varphi_\alpha^J(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha},$$

where $\varphi_\alpha^J = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i u_{J,i}^\alpha$. ($D_J$ is the total derivative operator describes in (5)).

Let us consider a one-parameter Lie group of infinitesimal transformations $(x, y, \bar{u}, \bar{v}, \bar{p})$ given by

$$\begin{aligned}
\widetilde{x} &= x + \varepsilon\xi_1(x,y,\bar{u},\bar{v},\bar{p}) + \mathcal{O}(\varepsilon^2), & \widetilde{y} &= y + \varepsilon\xi_2(x,y,\bar{u},\bar{v},\bar{p}) + \mathcal{O}(\varepsilon^2),\\
\widetilde{\bar{u}} &= \bar{u} + \varepsilon\eta_1(x,y,\bar{u},\bar{v},\bar{p}) + \mathcal{O}(\varepsilon^2), & \widetilde{\bar{v}} &= \bar{v} + \varepsilon\eta_2(x,y,\bar{u},\bar{v},\bar{p}) + \mathcal{O}(\varepsilon^2),\\
\widetilde{\bar{p}} &= \bar{p} + \varepsilon\eta_3(x,y,\bar{u},\bar{v},\bar{p}) + \mathcal{O}(\varepsilon^2), & &
\end{aligned}$$

where ε is the group parameter. Then one requires that this transformations leaves invariant the set of solutions of the system (2). This yields to the linear system of equations for the infinitesimals $\xi_1(x,y,\bar{u},\bar{v},\bar{p}), \xi_2(x,y,\bar{u},\bar{v},\bar{p})$, $\eta_1(x,y,\bar{u},\bar{v},\bar{p}), \eta_2(x,y,\bar{u},\bar{v},\bar{p})$ and $\eta_3(x,y,\bar{u},\bar{v},\bar{p})$. The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of

$$\mathbf{v} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial \bar{u}} + \eta_2\frac{\partial}{\partial \bar{v}} + \eta_3\frac{\partial}{\partial \bar{p}}.$$

This vector field has the second prolongation

$$\begin{aligned}
\mathbf{v}^{(2)} = \mathbf{v} &+ \varphi_1^x\frac{\partial}{\partial x} + \varphi_1^y\frac{\partial}{\partial y} + \varphi_1^{xx}\frac{\partial}{\partial \bar{u}_{xx}} + \varphi_1^{xy}\frac{\partial}{\partial \bar{u}_{xy}} + \varphi_1^{yy}\frac{\partial}{\partial \bar{u}_{yy}} + \varphi_2^x\frac{\partial}{\partial \bar{v}_x} + \varphi_2^y\frac{\partial}{\partial \bar{v}_y} + \varphi_2^{xx}\frac{\partial}{\partial \bar{v}_{xx}}\\
&+\varphi_2^{xy}\frac{\partial}{\partial \bar{v}_{xy}} + \varphi_2^{yy}\frac{\partial}{\partial \bar{v}_{yy}} + \varphi_3^x\frac{\partial}{\partial \bar{p}_x} + \varphi_3^y\frac{\partial}{\partial \bar{p}_y} + \varphi_3^{xx}\frac{\partial}{\partial \bar{p}_{xx}} + \varphi_3^{xy}\frac{\partial}{\partial \bar{p}_{xy}} + \varphi_3^{yy}\frac{\partial}{\partial \bar{p}_{yy}},
\end{aligned}$$

with the coefficients

$$\begin{aligned}
\varphi_1^x &= D_x(\varphi_1 - \xi_1\bar{u}_x - \xi_2\bar{u}_y) + \xi_1\bar{u}_{xx} + \xi_2\bar{u}_{xy}, & \varphi_1^y &= D_y(\varphi_1 - \xi_1\bar{u}_x - \xi_2\bar{u}_y) + \xi_1\bar{u}_{xy} + \xi_2\bar{u}_{yy},\\
\varphi_1^{xx} &= D_x^2(\varphi_1 - \xi_1\bar{u}_x - \xi_2\bar{u}_y) + \xi_1\bar{u}_{xxx} + \xi_2\bar{u}_{xxy}, & \varphi_1^{xy} &= D_xD_y(\varphi_1 - \xi_1\bar{u}_x - \xi_2\bar{u}_y) + \xi_1\bar{u}_{xxy} + \xi_2\bar{u}_{xxy},\\
\varphi_1^{yy} &= D_y^2(\varphi_1 - \xi_1\bar{u}_x - \xi_2\bar{u}_y) + \xi_1\bar{u}_{xxy} + \xi_2\bar{u}_{xyy}, & \varphi_2^x &= D_x(\varphi_2 - \xi_1\bar{v}_x - \xi_2\bar{v}_y) + \xi_1\bar{v}_{xx} + \xi_2\bar{v}_{xy},\\
\varphi_2^y &= D_y(\varphi_2 - \xi_1\bar{v}_x - \xi_2\bar{v}_y) + \xi_1\bar{v}_{xy} + \xi_2\bar{v}_{yy}, & \varphi_2^{xx} &= D_x^2(\varphi_2 - \xi_1\bar{v}_x - \xi_2\bar{v}_y) + \xi_1\bar{v}_{xxx} + \xi_2\bar{v}_{xxy},\\
\varphi_2^{xy} &= D_xD_y(\varphi_2 - \xi_1\bar{v}_x - \xi_2\bar{v}_y) + \xi_1\bar{v}_{xxy} + \xi_2\bar{v}_{xxy}, & \varphi_2^{yy} &= D_y^2(\varphi_2 - \xi_1\bar{v}_x - \xi_2\bar{v}_y) + \xi_1\bar{v}_{xxy} + \xi_2\bar{v}_{xyy},\\
\varphi_3^x &= D_x(\varphi_3 - \xi_1\bar{p}_x - \xi_2\bar{p}_y) + \xi_1\bar{p}_{xx} + \xi_2\bar{p}_{xy}, & \varphi_3^y &= D_y(\varphi_3 - \xi_1\bar{p}_x - \xi_2\bar{p}_y) + \xi_1\bar{p}_{xy} + \xi_2\bar{p}_{yy},\\
\varphi_3^{xx} &= D_x^2(\varphi_3 - \xi_1\bar{p}_x - \xi_2\bar{p}_y) + \xi_1\bar{p}_{xxx} + \xi_2\bar{p}_{xxy}, & \varphi_3^{xy} &= D_xD_y(\varphi_3 - \xi_1\bar{p}_x - \xi_2\bar{p}_y) + \xi_1\bar{p}_{xxy} + \xi_2\bar{p}_{xxy},\\
\varphi_3^{yy} &= D_y^2(\varphi_3 - \xi_1\bar{p}_x - \xi_2\bar{p}_y) + \xi_1\bar{p}_{xxy} + \xi_2\bar{p}_{xyy}, & &
\end{aligned}$$

where the operators D_x and D_y denote the total derivative with respect to x and y:

$$D_x = \frac{\partial}{\partial x} + \bar{u}_x \frac{\partial}{\partial \bar{u}} + \bar{v}_x \frac{\partial}{\partial \bar{v}} + \bar{p}_x \frac{\partial}{\partial \bar{p}} \cdots, \quad D_y = \frac{\partial}{\partial y} + \bar{u}_y \frac{\partial}{\partial \bar{u}} + \bar{v}_y \frac{\partial}{\partial \bar{v}} + \bar{p}_y \frac{\partial}{\partial \bar{p}} \cdots. \tag{5}$$

Using the invariance condition, i.e., applying the second prolongation $\mathbf{v}^{(2)}$, to system (2), and by solving the linear system

$$\begin{cases} \mathbf{v}^{(2)}\Big(\dfrac{\partial \bar{u}}{\partial x} + \dfrac{\partial \bar{v}}{\partial y}\Big) = 0, \\ \mathbf{v}^{(2)}\Big(\bar{u}\dfrac{\partial \bar{u}}{\partial x} + \bar{v}\dfrac{\partial \bar{v}}{\partial y} + \dfrac{1}{\rho}\dfrac{\partial \bar{p}}{\partial x} - \nu\dfrac{\partial^2 \bar{u}}{\partial y^2} + \dfrac{\partial}{\partial y}(\overline{u'v'})\Big) = 0, \qquad \pmod{(2)} \\ \mathbf{v}^{(2)}\Big(\dfrac{\partial \bar{p}}{\partial y}\Big) = 0, \end{cases}$$

the following system of 17 determining equations yields:

$$\begin{array}{llll} \xi_{1y} = 0, & \xi_{1\bar{u}} = 0, & \xi_{1\bar{v}} = 0, & \xi_{1\bar{v}} = 0, \\ \xi_{1xx} = 0, & \xi_{2x} = 0, & \xi_{2\bar{u}} = 0, & \xi_{2\bar{v}} = 0, \\ \xi_{2\bar{p}} = 0, & \xi_{2yy} = 0, & \eta_{1x} = 0, & \eta_{1y} = 0, \\ \eta_{1\bar{u}} = 0, & \eta_{1\bar{v}} = 0, & \eta_{1\bar{p}} = 2\xi_{1x} - 4\xi_{2y}, & \eta_2 = \bar{u}(\xi_{1x} - 2\xi_{2y}), \\ \eta_3 = -\xi_{2y}. & & & \end{array}$$

The solution of the above system gives the following coefficients of the vector field $\mathbf{v}$:

$$\xi_1 = C_1 + C_4 x, \quad \xi_2 = C_2 + C_5 y, \quad \eta_1 = C_4\bar{u} - 2C_5\bar{u}, \quad \eta_2 = -C_5\bar{v}, \quad \eta_3 = C_3 + 2C_4\bar{p} - 4C_5\bar{p},$$

where C_i for $i = 1, ..., 5$ are arbitrary constants, thus the Lie algebra $\mathfrak{g}$ of the Newtonian incompressible fluid's equations flow in turbulent boundary layers is spanned by the five vector fields

$$\begin{aligned} \mathbf{v}_1 &= \frac{\partial}{\partial x}, \\ \mathbf{v}_2 &= \frac{\partial}{\partial y}, \\ \mathbf{v}_3 &= \frac{\partial}{\partial \bar{p}}, \\ \mathbf{v}_4 &= x\frac{\partial}{\partial x} + \bar{u}\frac{\partial}{\partial \bar{u}} + 2\bar{p}\frac{\partial}{\partial \bar{p}}, \\ \mathbf{v}_5 &= y\frac{\partial}{\partial y} - 2\bar{u}\frac{\partial}{\partial \bar{u}} - \bar{v}\frac{\partial}{\partial \bar{v}} - 4\bar{p}\frac{\partial}{\partial \bar{p}}, \end{aligned}$$

The commutation relations between these vector fields is given by the table (1), where entry in row i and column j representing $[\mathbf{v}_i, \mathbf{v}_j]$.

The one-parameter groups G_i generated by the base of $\mathfrak{g}$ are given in the following table.

$$\begin{array}{ll} G_1 : (x + \varepsilon, y, \bar{u}, \bar{v}, \bar{p}), & G_2 : (x, y + \varepsilon, \bar{u}, \bar{v}, \bar{p}), \\ G_3 : (x, y, \bar{u}, \bar{v}, \bar{p} + \varepsilon), & G_4 : (xe^{\varepsilon}, y, \bar{u}e^{\varepsilon}, \bar{v}, \bar{p}e^{2\varepsilon}) \\ G_5 : (x, ye^{\varepsilon}, \bar{u}e^{-2\varepsilon}, \bar{v}e^{-\varepsilon}, \bar{p}e^{-4\varepsilon}) & \end{array}$$

Table 1
Commutation relations of $\mathfrak{g}$

$[\,,\,]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$	0
$\mathbf{v}_2$	0	0	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	0	0	0	$2\mathbf{v}_3$	$-4\mathbf{v}_3$
$\mathbf{v}_4$	$-\mathbf{v}_1$	0	$-2\mathbf{v}_3$	0	0
$\mathbf{v}_5$	0	$-\mathbf{v}_2$	$4\mathbf{v}_3$	0	0

Since each group G_i is a symmetry group and if $\bar{u} = f(x, y), \bar{v} = g(x, y)$ and $\bar{p} = h(x, y)$ are solutions of the system (2), so are the functions

$$\begin{array}{lllll}
\bar{u}_1 = f(x+\varepsilon, y), & \bar{u}_1 = f(x, y+\varepsilon), & \bar{u}_1 = f(x, y), & \bar{u}_1 = e^{-\varepsilon} f(xe^{\varepsilon}, y), & \bar{u} = e^{2\varepsilon} f(x, ye^{\varepsilon}), \\
\bar{v}_1 = g(x+\varepsilon, y), & \bar{v}_1 = g(x, y+\varepsilon), & \bar{v}_1 = g(x, y), & \bar{v}_1 = g(xe^{\varepsilon}, y), & \bar{v} = e^{\varepsilon} g(x, ye^{\varepsilon}), \\
\bar{p}_1 = h(x+\varepsilon, y), & \bar{p}_1 = h(x, y+\varepsilon), & \bar{p}_1 = h(x, y)+\varepsilon, & \bar{p}_1 = e^{-2\varepsilon} h(xe^{\varepsilon}, y), & \bar{p} = e^{4\varepsilon} h(x, ye^{\varepsilon}),
\end{array}$$

where ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination $c_1\mathbf{v}_1 + \cdots + c_1\mathbf{v}_5$ of the given vector fields. In particular if g is the action of the symmetry group near the identity, it can be represented in the form $g = \exp(\varepsilon\mathbf{v}_5) \circ \cdots \circ \exp(\varepsilon\mathbf{v}_1)$. Consequently, if $\bar{u} = f(x, y), \bar{v} = g(x, y)$ and $\bar{p} = h(x, y)$ be a solution for system (2), so is

$$\begin{aligned}
\bar{u} &= e^{\varepsilon} f\Big((x+\varepsilon)e^{\varepsilon}, (y+\varepsilon)e^{\varepsilon}\Big), \\
\bar{v} &= g\Big((x+\varepsilon)e^{\varepsilon}, (y+\varepsilon)e^{\varepsilon}\Big), \\
\bar{p} &= e^{2\varepsilon} h\Big((x+\varepsilon)e^{\varepsilon}, (y+\varepsilon)e^{\varepsilon}\Big) + \varepsilon e^{-\varepsilon},
\end{aligned} \tag{6}$$

a fact that can, of course, be checked directly.

3 Symmetry reduction for Newtonian incompressible fluid's equations flow in turbulent boundary layers

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation.

The first advantage of symmetry group method is to construct new solutions from known solutions. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the Eq. (2) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. The Newtonian incompressible fluid's equations flow in turbulent boundary layers expressed in the coordinates (x, y), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (r, s) corresponding to an infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries in the way of similarity transformations to leave invariant the system (2) under the group of symmetries. First we obtain the similarity variables for each term of the Lie algebra $\mathfrak{g}$, then we use this method to reduced the PDE and find the invariant solutions. Here our system has two-independent and four-dependent variables, thus, the similarity transformations and invariant transformations including of invariants up to order 2 comining in table (2) and (3), where * and ** are refer to ordinary invariants and first order differential invariants respectively.

Table 2
Reduction of system

vector field	symmetry transformations	similarities
$\mathbf{v}_1$	$r = x + \varepsilon, s = y, f(r,s) = u, g(r,s) = v, h(r,s) = p$	$x = s, y = r, u = f(r), v = g(r), p = h(r)$
$\mathbf{v}_2$	$r = x, s = y + \varepsilon, f(r,s) = u, g(r,s) = v, h(r,s) = p$	$x = r, y = s, u = f(r), v = g(r), p = h(r)$
$\mathbf{v}_3$	$r = x, s = y, f(r,s) = u, g(r,s) = v, h(r,s) = p + \varepsilon$	translation on p is the similarity
$\mathbf{v}_4$	$r = xe^{\varepsilon}, s = y, f(r,s) = e^{-\varepsilon}u, g(r,s) = v, h(r,s) = e^{-2\varepsilon}p$	$x = e^s, y = r, u = e^s f(r), v = g(r), p = e^{2s}h(r)$
$\mathbf{v}_5$	$r = x, s = ye^{\varepsilon}, f(r,s) = e^{2\varepsilon}u, g(r,s) = e^{\varepsilon}v, h(r,s) = e^{4\varepsilon}p$	$x = r, y = e^s, u = e^{-2s}f(r), v = e^{-s}g(r), p = e^{-4s}h(r)$

Table 3
Invariants

vector field	ordinary invariant	1st order diff. invariant	2nd order diff. invariant
$\mathbf{v}_1$	y, u, v, p	$*, u_x, v_x, p_x, u_y, v_y, p_y$	$**, u_{xx}, v_{xx}, p_{xx}, u_{xy}, v_{xy}, p_{xy}, u_{yy}, v_{yy}, p_{yy}$
$\mathbf{v}_2$	x, u, v, p	$*, u_x, v_x, p_x, u_y, v_y, p_y$	$**, u_{xx}, v_{xx}, p_{xx}, u_{xy}, v_{xy}, p_{xy}, u_{yy}, v_{yy}, p_{yy}$
$\mathbf{v}_3$	x, y, u, v	$*, u_x, v_x, p_x, u_y, v_y, p_y$	$**, u_{xx}, v_{xx}, p_{xx}, u_{xy}, v_{xy}, p_{xy}, u_{yy}, v_{yy}, p_{yy}$
$\mathbf{v}_4$	$u, \frac{u}{x}, v, \frac{p}{x^2}$	$*, u_x, xv_x, \frac{p_x}{x}, \frac{u_y}{x}, v_y, \frac{p_y}{x^2}$	$y, \frac{u}{x}, v, p, u_x, xv_x, \frac{p_x}{x}, \frac{u_y}{x}, v_y, \frac{p_y}{x^2}$, $xu_{xx}, x^2v_{xx}, p_{xx}, u_{xy}, xv_{xy}, \frac{p_{xy}}{x}, \frac{u_{yy}}{x}, v_{yy}, \frac{p_{yy}}{x^2}$
$\mathbf{v}_5$	x, y^2u, yv, y^4p	$*, y^2u_x, yv_x, y^4p_x, y^3u_y, y^2v_y, y^5p_y$	$*, y^2u_x, yv_x, y^4p_x, y^3u_y, y^2v_y, y^5v_y$, $y^2u_{xx}, yv_{xx}, y^4p_{xx}, y^3u_{xy}, y^2v_{xy}, y^5p_{xy}, y^4u_{yy}$, y^3v_{yy}, y^6p_{yy}

4 Characterization of invariant system of differential equations

Differential invariants help us to find general systems of differential equations which admit a prescribed symmetry group. One say, if G is a symmetry group for a system of PDEs with functionally differential invariants, then, the system can be rewritten in terms of differential invariants. For finding the differential invariants of the system (2) up to order 2, we should solve the following systems of PDEs:

$$\frac{\partial I}{\partial x} = 0, \quad \frac{\partial I}{\partial y} = 0, \quad \frac{\partial I}{\partial p} = 0, \quad x\frac{\partial I}{\partial x} + u\frac{\partial I}{\partial u} + 2p\frac{\partial I}{\partial p} = 0, \quad y\frac{\partial I}{\partial y} - 2u\frac{\partial I}{\partial u} - v\frac{\partial I}{\partial v} - 4p\frac{\partial I}{\partial p} = 0, \tag{7}$$

where I is a smooth function of (x, y, u, v, p),

$$\begin{aligned} &\frac{\partial I_1}{\partial x} = 0, \qquad \frac{\partial I_1}{\partial y} = 0, \qquad \frac{\partial I_1}{\partial p} = 0, \\ &x\frac{\partial I_1}{\partial x} + u\frac{\partial I_1}{\partial u} + 2p\frac{\partial I_1}{\partial p} - v_x\frac{\partial I_1}{\partial v_x} + \cdots + 2p_y\frac{\partial I_1}{\partial p_y} = 0, \\ &y\frac{\partial I_1}{\partial y} - 2u\frac{\partial I_1}{\partial u} - v\frac{\partial I_1}{\partial v} - 4p\frac{\partial I_1}{\partial p} - 2u_x\frac{\partial I_1}{\partial u_x} - \cdots - 5p_y\frac{\partial I_1}{\partial p_y} = 0, \end{aligned} \tag{8}$$

where I_1 is a smooth function of $(x, y, u, v, p, u_x, v_x, p_x, u_y, v_y, p_y)$,

$$\begin{aligned} &\frac{\partial I_2}{\partial x} = 0, \qquad \frac{\partial I_2}{\partial y} = 0, \qquad \frac{\partial I_2}{\partial p} = 0, \\ &x\frac{\partial I_2}{\partial x} + u\frac{\partial I_2}{\partial u} + 2p\frac{\partial I_2}{\partial p} - v_x\frac{\partial I_2}{\partial v_x} + \cdots + 2p_{yy}\frac{\partial I_2}{\partial p_{yy}} = 0, \\ &y\frac{\partial I_2}{\partial y} - 2u\frac{\partial I_2}{\partial u} - v\frac{\partial I_2}{\partial v} - 4p\frac{\partial I_2}{\partial p} - 2u_x\frac{\partial I_2}{\partial u_x} - \cdots - 6p_{yy}\frac{\partial I_2}{\partial p_{yy}} = 0. \end{aligned} \tag{9}$$

The solutions of (7), (8) and (9) are

$$
\begin{aligned}
&I = const.,\\
&I_1 = \Phi\Big(\frac{u_x}{v^2}, \frac{uv_x}{v^3}, \frac{p_x}{uv^2}, \frac{u_y}{uv}, \frac{v_y}{v^2}, \frac{p_y}{u^2v}\Big), && (10)\\
&I_2 = \Psi\Big(\frac{u_x}{v^2}, \frac{uv_x}{v^3}, \frac{p_x}{uv^2}, \frac{u_y}{uv}, \frac{v_y}{v^2}, \frac{p_y}{u^2v}, \frac{uu_{xx}}{v^4}, \frac{u^2v_{xx}}{v^5}, \frac{p_{xx}}{v_4}, \frac{u_{xy}}{v^3}, \frac{uv_{xy}}{v^4}, \frac{p_{xy}}{uv^3}, \frac{u_{yy}}{uv^2}, \frac{v_{yy}}{v^3}, \frac{p_{yy}}{u^2v^2}\Big), && (11)
\end{aligned}
$$

thus, the most general first and second order system (2) admitting (6) as a symmetry group can be rewritten with two arbitrary functions (10) and (11). Characterization of invariant system of differential equations is an alternative method for reduction of differential equations invariant under a symmetry group.

5 Optimal system of steady two-dimensional boundary-layer stagnation-point flow equations

Let a system of differential equation Δ admitting the symmetry Lie group G,be given. Now G operates on the set of solutions S of Δ. Let $s\cdot G$ be the orbit of s, and H be a subgroup of G. Invariant $H-$solutions $s \in S$ are characterized by equality $s \cdot S = \{s\}$. If $h \in G$ is a transformation and $s \in S$,then $h \cdot (s \cdot H) = (h \cdot s) \cdot (hHh^{-1})$. Consequently,every invariant $H-$solution s transforms into an invariant $hHh^{-1}-$solution (Proposition 3.6 of [5]).

Therefore, different invariant solutions are found from similar subgroups of G. Thus, classification of invariant $H-$solutions is reduced to the problem of classification of subgroups of G,up to similarity. An optimal system of $s-$dimensional subgroups of G is a list of conjugacy inequivalent $s-$dimensional subgroups of G with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s-$dimensional subalgebras forms an optimal system if every $s-$dimensional subalgebra of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$.

Let H and $\tilde{H}$ be connected, $s-$dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$ are conjugate subalgebras (Proposition 3.7 of [5]). Thus,the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and so we concentrate on it.

5.1 One-dimensional optimal system

For one-dimensional subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by a nonzero vector in Lie algebra symmetries of steady two-dimensional boundary-layer stagnation-point flow equations and so to "simplify" it as much as possible.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2}[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \tag{12}$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 5$. Let $F_i^\varepsilon : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i)\mathbf{v})$ is a linear map, for $i = 1, \cdots, 5$. The matrices M_i^ε of F_i^ε, $i = 1, \cdots, 5$, with respect to basis

$\{\mathbf{v}_1, \cdots, \mathbf{v}_5\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\varepsilon & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon & 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2\varepsilon & 1 & 0 \\ 0 & 0 & 4\varepsilon & 0 & 1 \end{pmatrix},$$
$$M_4^\varepsilon = \begin{pmatrix} e^\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{2\varepsilon} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_5^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & e^{-4\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

by acting these matrices on a vector field $\mathbf{v}$ alternatively we can show that a one-dimensional optimal system of $\mathfrak{g}$ is given by

$$X_1 = \mathbf{v}_3, \quad X_2 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2, \quad X_3 = \alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_3, \quad X_4 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3, \tag{14}$$

where α_i's are real constants.

5.2 Two-dimensional optimal system

Next step is to construct two-dimensional optimal system, i.e., classification of two-dimensional subalgebras of $\mathfrak{g}$. The process is by selecting one of the vector fields in (14), say, any vector field of (14). Let us consider X_1 (or $X_i, i = 2, 3, 4, 5$). Corresponding to it, a vector field $X = a_1\mathbf{v}_1 + \cdots + a_5\mathbf{v}_5$, where a_i's are smooth functions of (x, y, u, v, p) is chosen, so we must have

$$[X_1, X] = \lambda X_1 + \mu X, \tag{15}$$

the equation (15) leads us to the system

$$C^i_{jk}\alpha_j a_k = \lambda a_i + \mu\alpha_i \qquad (i = 1, 2, 3, 4, 5). \tag{16}$$

The solutions of the system (16), give one of the two-dimensional generator and the second generator is X_1 or, $X_i, i = 2, 3, 4, 5$ if selected. After the construction of all two-dimensional subalgebras, for every vector fields of (14), they need to be simplified by the action of (13) in the manner analogous to the way of one-dimensional optimal system.

Consequently the two-dimensional optimal system of $\mathfrak{g}$ has three classes of $\mathfrak{g}$'s members combinations such as

$$\begin{array}{llll} \mathbf{v}_1, \mathbf{v}_2, & \mathbf{v}_1, \mathbf{v}_3, & \mathbf{v}_2, \mathbf{v}_3, & \mathbf{v}_3, \mathbf{v}_4, \\ \mathbf{v}_1, \mathbf{v}_4, & \mathbf{v}_1, \mathbf{v}_5, & \mathbf{v}_2, \mathbf{v}_4 & \mathbf{v}_2, \mathbf{v}_5, \\ \mathbf{v}_3, \mathbf{v}_5, & \mathbf{v}_4, \mathbf{v}_5, & \mathbf{v}_1, \sum_{i=2}^5 \beta_i\mathbf{v}_i, & \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2, \mathbf{v}_3 + \beta_3(\mathbf{v}_4 + \mathbf{v}_5), \\ \beta_1\mathbf{v}_2 + \beta_2\mathbf{v}_3, \mathbf{v}_1 + \frac{5}{2}\beta_3(\mathbf{v}_4 + \mathbf{v}_5). & & & \end{array} \tag{17}$$

5.3 Three-dimensional optimal system

This system can be developed by the method of expansion of two-dimensional optimal system. For this take any two-dimensional subalgebras of (17), let us consider the first two vector fields of (17), and call them Y_1 and Y_2, thus, we have a subalgebra with basis $\{Y_1, Y_2\}$, find a vector field $Y = a_1\mathbf{v}_1 + \cdots + a_5\mathbf{v}_5$, where a_i's are smooth functions

of $(x, y, \bar{u}, \bar{v}, \bar{p})$, such the triple $\{Y_1, Y_2, Y\}$ generates a basis of a three-dimensional algebra. For that it is necessary an sufficient that the vector field Y satisfies the equations

$$\begin{aligned} &[Y_1, Y] = \lambda_1 Y + \mu_1 Y_1 + \nu_1 Y_2, \\ &[Y_2, Y] = \lambda_2 Y + \mu_2 Y_1 + \nu_2 Y_2, \end{aligned} \tag{18}$$

and following from (18), we obtain the system

$$\begin{aligned} &C^i_{jk} \beta^j_r a_k = \lambda_1 a_i + \mu_1 \beta^i_r + \nu_1 \beta^i_s, \qquad && \alpha = 1, 2, 3, 4, 5 \\ &C^i_{jk} \beta^j_s a_k = \lambda_2 a_i + \mu_2 \beta^i_r + \nu_2 \beta^i_s, \qquad && \alpha = 1, 2, 3, 4, 5. \end{aligned} \tag{19}$$

The solutions of system (19) is linearly independent of $\{Y_1, Y_2\}$ and give a three-dimensional subalgebra. This process is used for the another two couple vector fields of (17).

Consequently the three-dimensional optimal system of $\mathfrak{g}$ is given by

$$\begin{aligned} &\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \quad \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \quad \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5, \quad \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \quad \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5, \\ &\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \quad \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \quad \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \quad \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5, \quad \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5. \end{aligned} \tag{20}$$

5.4 Four-dimensional optimal system

Four-dimensional optimal system can be developed by the method of expansion of three-dimensional optimal system analogous to three-dimensional. First step is selecting a three-dimensional subalgebra of (20), and call them Z_1, Z_2 and Z_3, then find a vector field $Z = a_1 \mathbf{v}_1 + \cdots + a_5 \mathbf{v}_5$, where a_i's are smooth functions of $(x, y, \bar{u}, \bar{v}, \bar{p})$, such the foursome $\{Z_1, Z_2, Z_3, Z\}$ generates a basis of a four-dimensional algebra. It is necessary an sufficient that the vector field Z satisfies the equations

$$\begin{aligned} &[Z_1, Z] = \lambda_1 Z + \mu_1 Z_1 + \nu_1 Z_2 + \gamma_1 Z_3, \\ &[Z_1, Z] = \lambda_2 Z + \mu_2 Z_1 + \nu_2 Z_2 + \gamma_2 Z_3, \\ &[Z_1, Z] = \lambda_3 Z + \mu_3 Z_1 + \nu_3 Z_2 + \gamma_3 Z_3, \end{aligned} \tag{21}$$

following from (21), we obtain the system

$$\begin{aligned} &C^i_{jk} a_k = \lambda_1 a_i + \mu_1 + \nu_1 + \gamma_1, \qquad && \alpha = 1, 2, 3, 4, 5, \\ &C^i_{jk} a_k = \lambda_2 a_i + \mu_2 + \nu_2 + \gamma_2, \qquad && \alpha = 1, 2, 3, 4, 5, \\ &C^i_{jk} a_k = \lambda_3 a_i + \mu_3 + \nu_3 + \gamma_3, \qquad && \alpha = 1, 2, 3, 4, 5. \end{aligned} \tag{22}$$

The solutions of system (22) is linearly independent of $\{Z_1, Z_2, Z_3\}$ and give a four-dimensional subalgebra. This process is used for the another triple vector fields of (20). All previous calculations lead to the table (4) for the optimal system of $\mathfrak{g}$.

6 Lie Algebra Structure

$\mathfrak{g}$ has a no any non-trivial *Levi decomposition* in the form of $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_1$, because $\mathfrak{g}$ has no any non-trivial radical, i.e., if $\mathfrak{r}$ be the radical of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{r}$.

If we want to integration an involuting distribution, the process decomposes into two steps:

- integration of the evolutive distribution with symmetry Lie algebra $\mathfrak{g}/\mathfrak{r}$, and
- integration on integral manifolds with symmetry algebra $\mathfrak{r}$.

Table 4
Optimal system of subalgebras

dimension	1	2	3	4	5
	$\langle \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \rangle$
subalgebras	$\langle \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5 \rangle$	
	$\langle \alpha_1 \mathbf{v}_2 + \alpha_2 \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \rangle$	
	$\langle \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \rangle$	
		$\langle \mathbf{v}_3, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \rangle$	
		$\langle \mathbf{v}_1, \mathbf{v}_4 \rangle$	$\langle \mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5 \rangle$		
		$\langle \mathbf{v}_1, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$		
		$\langle \mathbf{v}_2, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5 \rangle$		
		$\langle \mathbf{v}_3, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \rangle$		
		$\langle \mathbf{v}_4, \mathbf{v}_5 \rangle$	$\langle \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \rangle$		
		$\langle \mathbf{v}_1, \sum_{i=2}^{5} \beta_i \mathbf{v}_i \rangle$			
		$\langle \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2, \mathbf{v}_3 + \beta_3 (\mathbf{v}_4 + \mathbf{v}_5) \rangle$			
		$\langle \beta_1 \mathbf{v}_2 + \beta_2 \mathbf{v}_3, \mathbf{v}_1 + \frac{5}{2} \beta_3 (\mathbf{v}_4 + \mathbf{v}_5) \rangle$			

First, applying this procedure to the radical $\mathfrak{r}$ we decompose the integration problem into two parts: the integration of the distribution with semisimple algebra $\mathfrak{g}/\mathfrak{r}$, then the integration of the restriction of distribution to the integral manifold with the solvable symmetry algebra $\mathfrak{r}$.

The last step can be performed by quadratures. Moreover, every semisimple Lie algebra $\mathfrak{g}/\mathfrak{r}$ is a direct sum of simple ones which are ideal in $\mathfrak{g}/\mathfrak{r}$. Thus, the Lie-Bianchi theorem reduces the integration problem to involutive distributions equipped with simple algebras of symmetries. Thus, integrating of system (2), become so much easy.

The Lie algebra $\mathfrak{g}$ is solvable and non-semisimple. It is solvable because if $\mathfrak{g}^{(1)} = \langle \mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j] \rangle = [\mathfrak{g}, \mathfrak{g}]$, we have $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \langle \mathbf{v}_1, \cdots, \mathbf{v}_5 \rangle$, and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \langle \mathbf{v}_1, \mathbf{v}_2, 2\mathbf{v}_3 \rangle$, so, we have a chain of ideals $\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \{0\}$, and it is non-semisimple because its killing form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -8 \\ 0 & 0 & 0 & -8 & 17 \end{pmatrix},$$

is degenerate.

According to the table of the commutators, $\mathfrak{g}$ has two abelian 2 and 3-dimensional subalgebras spanned by $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ and $\langle \mathbf{v}_4, \mathbf{v}_5 \rangle$ respectively such that the first one is an ideal in $\mathfrak{g}$, so, we can decompose $\mathfrak{g}$ in the to semidirect sum of $\mathfrak{g} = \mathbb{R}^3 \ltimes \mathbb{R}^2$.

7 Conclusion

In this article group classification of Newtonian incompressible fluid's equations flow in turbulent boundary layers and the algebraic structure of the symmetry group is considered. Classification of r-dimensional subalgebra is determined by constructing r-dimensional optimal system. Some invariant objects are fined and the Lie algebra structure of symmetries is found.

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